DEGENERATION OF A COMPACT RIEMANN SURFACE OF GENUS 2

BY

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ABSTRACT

A problem in the degeneration of a compact, two-sheeted, Riemann surface of genus 2 is studied, using theta function techniques. The three moveable branch points coalesce to the fourth branch point on the first limiting surface while the triangles formed by these points are all essentially similar. Applying a conformal map, we see that these points represent the finite branch points on the second of the limiting surfaces.

1. Introduction

In this paper we study the degeneration of a compact Riemann surface of genus 2. We take a specific concrete two-sheeted realization of the surface with a particular homology basis. The surface is then split according to a given prescription (see section 4). We obtain the beautiful result that the three branch points (see section 3) coalesce and also the fact, that as they come together, the triangles they form are all essentially similar.

In genus one, there are classical results concerning the relationship between the fundamental domains of the λ-plane under the group, $G(\lambda)$, of the six Moebius transformations of the λ-plane onto itself, and the Siegel upper half-plane, $\mathfrak{S}_1$, under the group of fractional linear transformations,

$$\tau \to \frac{a\tau + b}{c\tau + d},\ a, b, c, d \text{ integers and } \det \begin{vmatrix} d & c \\ b & a \end{vmatrix} = +1.$$

The classical result states that the relationship between the λ and τ planes is given by the fact that the fundamental domain of the modular group is mapped 1-1 conformally, except for vertices, onto the fundamental domain of $G(\lambda)$ (see [5]).

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The present paper is the result of an exploration into the extension to genus two of the results known in genus one. What is found here is a relationship between a boundary point, of points corresponding to Riemann surfaces, in the Siegel upper half-plane $\mathfrak{S}_2$, and points in C^3, where each copy of C is slit from $-\infty$ to 0 and from 1 to ∞.

We obtain results about the degeneration of a specific Riemann surface by defining the degeneration in terms of the degeneration of a period matrix. In [2] a prescription was given to define a "splitting of the surface", and results were obtained concerning the period matrix of the degenerating surface. Both surfaces were "seen" before the degeneration given in [2]. In this paper we apparently see the first Riemann surface and not the second. By means of a conformal map we are able to view the degeneration in such a way that we see the second surface rather than the first (see section 5).

2.

DEFINITION 1. A g-characteristic (g an integer $\geqq 1$) is a $2 \times g$ matrix of integers $\left[{\varepsilon \atop \varepsilon'}\right]$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_g)$ and $\varepsilon' = (\varepsilon'_1, \varepsilon'_2, \cdots, \varepsilon'_g)$. The characteristic is said to be even or odd depending on whether $\sum_{i=1}^{g} \varepsilon_i \varepsilon'_i$ is even or odd. A reduced g-characteristic has only 0's and 1's as its entries. A reduced g-characteristic is obtained from a given g-characteristic by replacing each entry by its least non-negative residue mod 2. A g-characteristic and its reduced representative are even or odd together.

DEFINITION 2. Let $\zeta = (\zeta_1, \cdots, \zeta_g)$ be a complex g-vector and $T = (t_{ij})$ be a $g \times g$ symmetric matrix with positive definite imaginary part. The set of all such matrices is the generalized upper half-plane, $\mathfrak{S}_g$, known as the Siegel upper half-plane of genus g. The first order theta function with g-characteristic, $\left[{\varepsilon \atop \varepsilon'}\right]$, is defined by the following series, which converges absolutely and uniformly on compact subsets of $C^g \times \mathfrak{S}_g$:

$$\theta\begin{bmatrix}\varepsilon \\ \varepsilon'\end{bmatrix}(\zeta, T) = \sum_{n_1, n_2, \cdots, n_g = -\infty}^{\infty} \exp \pi i \left\{ \sum_{i,j=1}^{g} t_{ij}\left(n_i + \frac{\varepsilon_i}{2}\right)\left(n_j + \frac{\varepsilon_j}{2}\right) + 2\sum_{i=1}^{g}\left(n_i + \frac{\varepsilon_i}{2}\right)\left(\zeta_i + \frac{\varepsilon'_i}{2}\right)\right\}.$$

REMARK. It is clear from the definition that the theta function splits when T splits into blocks along the major diagonal.

The theta constant with g-characteristic $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ at T is

$$\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (0, T).$$

Where no confusion can arise, we sometimes shorten the expression for the theta constant to $\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$.

We state now some properties of the theta functions in the form of a series of lemmas. Proofs can be found in [1].

LEMMA 1. *The first order theta function* $\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (\zeta, T)$ *is an even or odd function of* ζ *depending on whether* $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ *is an even or odd characteristic.*

As an immediate consequence of Lemma 1, we have the fact that all theta constants with odd g-characteristics are zero.

LEMMA 2. (*Reduction formula*). *If* $\varepsilon = \hat{\varepsilon} + 2V$ *and* $\varepsilon' = \hat{\varepsilon}' + 2V'$, where V *and* V' *are integral g-vecters, then*

$$\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (\zeta, T) = (-1)^{\sum_{i=1}^{g} \varepsilon_i v_i'}\, \theta \begin{bmatrix} \hat{\varepsilon} \\ \hat{\varepsilon}' \end{bmatrix} (\zeta, T).$$

DEFINITION 3. The period matrix of the theta functions with characteristics and matrix T is the $g \times 2g$ matrix $(I_g | \, T)$ whose left half is the $g \times g$ identity matrix I_g and whose right half is T. A period is an integral linear combination of the columns of the period matrix, i.e.,

$$\begin{Bmatrix} \mu \\ \mu' \end{Bmatrix} = \mu_1' e^{(1)} + \cdots + \mu_g' e^{(g)} + \mu_1 t^{(1)} + \cdots + \mu_g t^{(g)},$$

where $e^{(i)}$ and $t^{(i)}$ are the respective ith columns of I_g and T. Thus

$$\begin{Bmatrix} \mu \\ \mu' \end{Bmatrix} = \left(\mu_1' + \sum_i \mu_i t_{1i}, \cdots, \mu_g' + \sum_i \mu_i t_{gi} \right).$$

DEFINITION 4. A half-period, $\begin{pmatrix} \mu \\ \mu' \end{pmatrix}$, is literally half a period, i.e.,

$$\begin{pmatrix} \mu \\ \mu' \end{pmatrix} = \frac{1}{2} \begin{Bmatrix} \mu \\ \mu' \end{Bmatrix}.$$

LEMMA 3. (*Functional equation*).

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\left(\zeta+\begin{Bmatrix}\mu\\ \mu'\end{Bmatrix},T\right)=\exp\pi i\left\{\sum_i(\varepsilon_i\mu_i'-\varepsilon_i'\mu_i)-2\sum_i\mu_i\zeta_i-\sum_{i\,j}t_{ij}\mu_i\mu_j\right\}\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\zeta,T).$$

LEMMA 4. *(Substitution formula).*

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\left(\zeta+\begin{pmatrix}\mu\\ \mu'\end{pmatrix},T\right)=\exp\pi i\left\{-\frac{1}{4}\sum t_{ij}\mu_i\mu_j-\frac{1}{2}\sum\mu_i(\varepsilon_i'+\mu_i')-\sum_i\mu_i\zeta_i\right\}\times$$
$$\theta\begin{bmatrix}\varepsilon+\mu\\ \varepsilon'+\mu'\end{bmatrix}(\zeta,T).$$

Now let S be a compact Riemann surface of genus g and let $\gamma_1,\cdots,\gamma_g,\delta_1,\cdots,\delta_g$ be a canonical homology basis for S. Let $d\zeta_1,\cdots,d\zeta_g$ be the normal basis of Abelian differentials of first kind on S with respect to the given homology basis, i.e., $\int_\gamma d\zeta_i=\delta_{ij}$. Then it is well known that the matrix $\pi=(\pi_{ij})$, when $\pi_{ij}=\int_{\delta_j}d\zeta_i$ is an element of $\mathfrak{S}_g$.

DEFINITION 5. The Riemann theta function with characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ associated with S, $\gamma_1,\cdots,\gamma_g$, $\delta_1,\cdots,\delta_g$ is $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\zeta,\pi)$ and the associated Riemann theta constant is $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,\pi)$.

We define a map of $S\to C^g$ by $\zeta(P)=(\int_{P_0}^P d\zeta_1,\int_{P_0}^P d\zeta_2,\cdots,\int_{P_0}^P d\zeta_g)$ where P_0 is a fixed point (the base point) on S. This map may be extended to a map of divisors on $S\to C^g$ by setting $\zeta(P_1^{\alpha_1}\cdots P_r^{\alpha_r})=\alpha_1\zeta(P_1)+\cdots+\alpha_r\zeta(P_r)$. By means of $\zeta(P)$ we may derive from the Riemann theta function, with any characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$, associated with S a multivalued function on S.

LEMMA 5. $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\zeta(P),\pi)$ *is either identically zero on S or else it has divisor of zeros* $P_1\cdots P_g$ *such that*

$$\zeta(P_1\cdots P_g)+K=\begin{pmatrix}\varepsilon\\ \varepsilon'\end{pmatrix}+\begin{Bmatrix}\mu\\ \mu'\end{Bmatrix}$$

for some period $\begin{Bmatrix}\mu\\ \mu'\end{Bmatrix}$, *where K is a vector of constants depending on* $\gamma_1,\cdots,\gamma_g,\delta_1,\cdots,\delta_g$ and P_0. *The vector K is called the vector of Riemann constants.*

3.

Let S be a compact Riemann surface of genus 2. S always admits a representation as the concrete Riemann surface of $w^2 = P_6(z) = (z - z_1)(z - z_2)\cdots(z-z_6)$ for suitable $z_1, z_2, \cdots, z_6$. Furthermore, we can normalize so that three of the six branch points, z_i, are at 0, 1, and ∞. The equation takes the form $w^2 = z(1-z)(1-\lambda_1 z)(1-\lambda_2 z)(1-\lambda_3 z)$, and the Riemann surface has a concrete realization as a two-sheeted cover of the sphere with distinct branch points over $z = 0$, $z = 1$, $z = 1/\lambda_1$, $z = 1/\lambda_2$, $z = 1/\lambda_3$, and $z = \infty$.

LEMMA 6. *If a hyperelliptic Riemann surface is represented by the equation*

$$w^2 = z(1-z)(1-\lambda_1 z)(1-\lambda_2 z)\cdots(1-\lambda_{2g-1} z)$$

so that it has distinct branch points over $0, 1, 1/\lambda_1, \cdots, 1/\lambda_{2g-1}, \infty$ *one may assume that its real branch points other than* 0 *and* 1 (*if any*) *are all greater than one and in ascending order.*

PROOF. The result is achieved by applying some auxiliary transformations. See [4].

With Lemma 6 in mind, one may, without loss of generality, assume that the concrete realization of S is as shown in Fig. 1. We then draw the particular homology basis shown in Fig. 1. Note that α is not part of the homology basis.

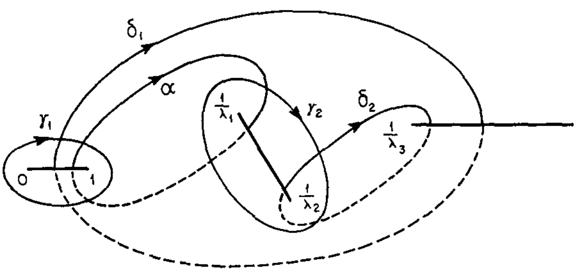

Fig. 1.

We will obtain the following:

THEOREM 1. *Let S be as described in Fig.* 1. *Then the branch points* $1/\lambda_i$, $i = 1,2,3$ *are computed as the following quotients of products of Riemann theta constants*:

$$\frac{1}{\lambda_1} = \frac{\theta^2\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(0,\pi)\,\theta^2\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}(0,\pi)}{\theta^2\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}(0,\pi)\,\theta^2\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}(0,\pi)},$$

$$\frac{1}{\lambda_2} = \frac{\theta^2\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}(0,\pi)\,\theta^2\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}(0,\pi)}{\theta^2\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}(0,\pi)\,\theta^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}(0,\pi)},$$

$$\frac{1}{\lambda_3} = \frac{\theta^2\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(0,\pi)\,\theta^2\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}(0,\pi)}{\theta^2\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}(0,\pi)\,\theta^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}(0,\pi)}.$$

An important point is the observation that the differentials of first kind on S are linear combinations of dz/w and zdz/w. Hence every differential of first kind, in particular, $d\zeta_1$, $d\zeta_2$ the normal differentials with respect to $\gamma_1, \gamma_2, \delta_1, \delta_2$, change sign under sheet interchange $(z, w) \to (z, -w)$. Each basis cycle is homologous to a rectilinear path between appropriate pairs of branch points run through in one direction on one sheet of S and back on the other. One immediately deduces from these facts that the integral of the vector differential $(d\zeta_1, d\zeta_2)$ between any two branch points is a half-period in the sense of Definition 4.

We obtain Table I where we have taken 0 as the base point of the integrals.

TABLE I

$\zeta(0) = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$	
$\zeta(1) = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$	where the path of integration is taken along the top sheet on the top of the cut $0 \to 1$.
$\zeta\left(\frac{1}{\lambda_1}\right) = \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}$	where we proceed from 1 on the top sheet to $1/\lambda_1$.
$\zeta\left(\frac{1}{\lambda_2}\right) = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$	where we proceed from $1/\lambda_1$ on the top sheet on the left of the branch cut as we go to $1/\lambda_2$.
$\zeta\left(\frac{1}{\lambda_3}\right) = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$	where we proceed on the top sheet from $1/\lambda_2$ to $1/\lambda_3$.
$\zeta(\infty) = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$	where we proceed along the top sheet on the top of the cut from $1/\lambda_3$ to ∞.

We shall illustrate how to compute the half periods. $\zeta(1/\lambda_1)$ is the first one which might present some difficulty. Let α be the cycle shown in Fig. 1. The intersection number of α with γ_1 is -1 and with γ_2 is $+1$. Furthermore, α has zero intersection with the other basis cycles. Now $\alpha = a\gamma_1 + b\gamma_2 + c\delta_1 + d\delta_2$ and we need only compute a, b, c, and d. Using the information above concerning the intersection numbers of α with the basis cycles, we obtain $a = b = 0$ $c = +1$, $d = -1$. The rest is clear and is left to the reader.

Definition 6. A multiplicative function with characteristic $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ on S is a two-valued function f, which is meromorphic on S and is multiplied by $(-1)^{\varepsilon_i}$ ($(-1)^{\varepsilon'_i}$ resp.) when analytically continued around γ_i (δ_i respectively).

Lemma 7. *On the Riemann surface S, with the canonical homology basis shown in Fig.* 1, *we have Table II of multiplicative functions with their characteristics and zeros and poles at the indicated branch points.*

Table II

Function	*Characteristic*	*Zero*	*Pole*
$\sqrt{z}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	0	∞
$\sqrt{1-\lambda_1 z}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\frac{1}{\lambda_1}$	∞
$\sqrt{1-\lambda_2 z}$	$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$	$\frac{1}{\lambda_2}$	∞
$\sqrt{1-\lambda_3 z}$	$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$	$\frac{1}{\lambda_3}$	∞

Moreover,

Table III

$$\sqrt{z} = -\frac{\theta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}(0,\pi)\,\theta\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}(\zeta(P),\pi)}{\theta\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}(0,\pi)\,\theta\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}(\zeta(P),\pi)} = \frac{\theta\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}(0,\pi)\,\theta\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}(\zeta(P),\pi)}{\theta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(0,\pi)\,\theta\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}(\zeta(P),\pi)}$$

$$\sqrt{1-\lambda_1 z} = \frac{\theta\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}(0,\pi)\,\theta\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(\zeta(P),\pi)}{\theta\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(0,\pi)\,\theta\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}(\zeta(P),\pi)} = \frac{\theta\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}(0,\pi)\,\theta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(\zeta(P),\pi)}{\theta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(0,\pi)\,\theta\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}(\zeta(P),\pi)}$$

$$\sqrt{1-\lambda_2 z} = \frac{\theta\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}(\zeta(P),\pi)}{\theta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}(\zeta(P),\pi)} = \frac{\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}(\zeta(P),\pi)}{\theta\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}(\zeta(P),\pi)}$$

$$\sqrt{1-\lambda_3 z} = \frac{\theta\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(\zeta(P),\pi)}{\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}(\zeta(P),\pi)} = \frac{\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}(\zeta(P),\pi)}{\theta\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}(\zeta(P),\pi)}$$

PROOF. First of all we note that for these multiplicative functions we have more than one representation as a quotient of theta functions thus showing that their theta function representations are not unique.

The multiplicative character of each of the various functions and their zeros and poles is obvious (Lemma 3, T replaced by π); hence we prove only the results in Table III.

The reader can easily verify that the given quotients have the proper characteristics. There remains to show, therefore, that the theta quotients have the proper zeros and poles and that the normalization given by the quotient of theta constants is correct. In this connection the reader is referred to [4] where similar computations are carried out.

We are now ready to turn to the proof of Theorem 1. First of all, in the first equation of Table III, set $P = 1/\lambda_1$ to find:

$$\sqrt{\frac{1}{\lambda_1}} = -\frac{\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(0)\,\theta\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}(\zeta(1/\lambda_1))}{\theta\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}(0)\,\theta\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}(\lambda(1/\lambda_1))}$$

which equals, using Table I,

$$-\frac{\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(0)\,\theta\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\left(\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\right)}{\theta\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}(0)\,\theta\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\left(\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\right)}$$

which equals, by Lemma 4,

$$-\frac{\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}(0)\,\theta\begin{bmatrix}2 & -1\\ 2 & 0\end{bmatrix}(0)}{\theta\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}(0)\,\theta\begin{bmatrix}1 & -1\\ 2 & 0\end{bmatrix}(0)} \times \frac{\exp \pi i - \frac{1}{2}(2)}{\exp \pi i - \frac{1}{2}(2)}$$

which equals, by Lemma 2,

$$\frac{\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}0&1\\0&0\end{bmatrix}(0,\pi)}{\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}1&1\\0&0\end{bmatrix}(0,\pi)}.$$

Next, in the same equation, set $P = 1/\lambda_3$ to find, by a computation similar to the one above

$$\sqrt{\frac{1}{\lambda_3}} = \frac{\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}(0,\pi)}{\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}(0,\pi)}.$$

To obtain $1/\lambda_2$ we use the extreme right hand side of the first line in Table III and an analysis similar to what we used for $1/\lambda_1$ and $1/\lambda_3$ and we obtain

$$\sqrt{\frac{1}{\lambda_2}} = \frac{\theta\begin{bmatrix}0&1\\0&0\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}(0,\pi)}{\theta\begin{bmatrix}1&1\\0&0\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}(0,\pi)}.$$

This concludes Theorem 1.

4.

We are now prepared to turn to the problem of the degeneration of the Riemann surface, S. In order to define what we mean by the degeneration we will need some further auxiliary definitions.

DEFINITION 7. The homogeneous Siegel modular group, $S_p(g,z)$ of degree g, is the set of $2g \times 2g$ matrices M, with integral entries satisfying

$$M\begin{bmatrix}O_g & I_g\\ -I_g & O_g\end{bmatrix}M^T = \begin{bmatrix}D & C\\ B & A\end{bmatrix}\begin{bmatrix}O_g & I_g\\ -I_g & O_g\end{bmatrix}\begin{bmatrix}D & C\\ B & A\end{bmatrix}^T = \begin{bmatrix}O_g & I_g\\ -I_g & O_g\end{bmatrix}$$

where T denotes transpose, O_g and I_g are respectively the $g \times g$ zero and unit matrices, and A, B, C, D are $g \times g$ matrices in the indicated positions.

DEFINITION 8. The inhomogeneous Siegel modular group $\mathcal{M}^g$ is isomorphic to $S_p(g,z)/\{I_{2g}, -I_{2g}\}$ and acts on $\mathfrak{S}_g$ by $(\pi'_{ij}) = (A\pi + B(C\pi + D)^{-1} = M\cdot(\pi_{ij})$, where (π_{ij}) and (π'_{ij}) belong to $\mathfrak{S}_g$ and the operations are matrix operations. Two matrices are said to be equivalent if they are related by an element of $\mathcal{M}^g$. The elements of $\mathcal{M}^g$ which are congruent to the identity mod 2 are denoted by $\mathcal{M}^g_2$.

Let (π_{ij}) be a given element in $\mathfrak{S}_2$ which is not equivalent to a diagonal matrix. We keep the elements π_{11} and π_{22} fixed and allow π_{12} to go to zero. We restrict

π_{12} by demanding that, as it goes to zero, we are at no time at a matrix which is equivalent to a diagonal one. That we can do this is clear. Indeed, such matrices have lower complex dimension than the full space, $\mathfrak{S}_2$. There exist Riemann surfaces of genus 1 with period ratios π_{11} and π_{22}. Thus we associate with π_{11} and π_{22}, Riemann surfaces of genus 1, S_1 and S_2, respectively (see [5]).

DEFINITION 9. We define an admissible splitting degeneration of the Riemann surface S, with period matrix (π_{ij}) as follows: Let π_{12} tend to zero as described above and we obtain in the limit the two surfaces S_1 and S_2, that is, S has split into S_1 and S_2. We choose in such π_{12} a way that it is close to zero and the three moveable branch points are all far away from the interval $[0,1]$.

THEOREM 2. *Let S be as in Fig.* 1. *Let S degenerate as in Definition* 9. *Then the three branch points, $1/\lambda_1$, $1/\lambda_2$, $1/\lambda_3$ on S all tend to the same point, $1/\lambda$. In particular the point*

$$\frac{1}{\lambda} = \frac{\theta^4\begin{bmatrix}0\\0\end{bmatrix}(0,\pi_{11})}{\theta^4\begin{bmatrix}1\\0\end{bmatrix}(0,\pi_{11})},$$

where $1/\lambda$ is the branch point on the surface S_1, i.e., the quotient of theta constants computes the fourth branch point on S_1 or one of its cross-ratios (*see Introduction*).

PROOF. The result is clear from the remark after Definition 2 concerning the splitting of the theta function. Indeed, the theta functions converge absolutely and uniformly for matrices in $\mathfrak{S}_2$ and it is known [3], that on a surface of genus 2, no even theta vanishes. Furthermore, none of the limiting theta functions vanish in genus one. That the quotient of theta constants gives the formula for $1/\lambda$ on S_1 is a result which can be obtained using machinery in genus one similar to the computations we have produced here in genus 2. The reader can find those computations in [5].

See the proof of Theorem 3 for computations showing how to expand the quotients of theta constants in powers of π_{12}.

REMARK. If the entries in the matrix are all pure imaginary and the degeneration is taken through purely imaginary values only, then the branch points are all real, given in ascending order, and tend to one another along the real axis.

THEOREM 3. *Let S be as in Fig.* 1. *The three branch points $1/\lambda_1$, $1/\lambda_2$, $1/\lambda_3$*

form a triangle. Let S degenerate as in Definition 9. The triangles formed by the three branch point vary as the points coalesce. These triangles are all essentially similar, that is, the angles formed by the three points are almost constant.

In order to prove this theorem we need

LEMMA 8. *Let S be as described in Fig. 1. Then*

TABLE IV

$$\sqrt{\frac{\lambda_2-\lambda_1}{\lambda_2}} = -i\,\frac{\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\theta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}{\theta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}},\quad \sqrt{\frac{\lambda_3-\lambda_2}{\lambda_3}} = -i\,\frac{\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\theta\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}}{\theta\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\theta\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}},$$

$$\sqrt{\frac{\lambda_3-\lambda_1}{\lambda_3}} = -i\,\frac{\theta\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}}{\theta\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\theta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}},\quad \sqrt{\frac{\lambda_1-\lambda_3}{\lambda_1}} = \frac{\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\theta\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}}{\theta\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\theta\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}},$$

$$\sqrt{\frac{\lambda_1-\lambda_2}{\lambda_1}} = \frac{\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\theta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}{\theta\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\theta\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}}.$$

Note that we suppress the dependence of the theta constants on π.

PROOF. In order to get the entries in Table IV we use the results from Table III, Lemma 7. We will compute the first entry and leave the rest to the reader.

In the second row of Table III, on the right hand side, set P equal to $1/\lambda_2$. We obtain

$$\sqrt{\frac{\lambda_2-\lambda_1}{\lambda_2}} = \frac{\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}(\zeta(1/\lambda_2),\pi)}{\theta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}(0,\pi)\,\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}(\zeta(1/\lambda_2),\pi)}.$$

From Table I, $\zeta(1/\lambda_2) = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$. Substituting in the above equation, we get

$$\sqrt{\frac{\lambda_2-\lambda_1}{\lambda_2}} = \frac{\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}(0)\,\theta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\right)}{\theta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}(0)\,\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\left(\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\right)},$$

which equals, by Lemma 4,

$$\frac{\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}(0)\theta\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}(0)\exp\pi i - \tfrac{1}{2}(-1)}{\theta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}(0)\theta\begin{bmatrix}2 & 0\\ 2 & 2\end{bmatrix}(0)},$$

which equals, by Lemma 2,

$$-\,i\,\frac{\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\theta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}{\theta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\theta\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}}.$$

The rest of the results in Table IV are obtained using various entries from Table III.

We are now in a position to prove Theorem 3.

Consider, for example, $((1/\lambda_1) - (1/\lambda_3))/((1/\lambda_1) - (1/\lambda_2))$. This expression equals $(\lambda_3 - \lambda_1)/(\lambda_2 - \lambda_1) \times (\lambda_2/\lambda_3)$. Using the results of Table IV we find that

$$\frac{(1/\lambda_1)-(1/\lambda_3)}{(1/\lambda_1)-(1/\lambda_2)} = \frac{\theta^2\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\theta^2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}}{\theta^2\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\theta^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}} \times \frac{\theta^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\theta^2\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}}{\theta^2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\theta^2\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}.$$

Thus,

$$\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_3}\right) = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) \frac{\theta^2\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\theta^2\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}}{\theta^2\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\theta^2\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}.$$

When $\pi_{12} = 0$ the quotient of theta constants in the last expression splits as follows (here we give each argument):

$$\frac{\theta^2\begin{bmatrix}0\\ 1\end{bmatrix}(0,\pi_{11})\,\theta^2\begin{bmatrix}0\\ 0\end{bmatrix}(0,\pi_{22})\,\theta^2\begin{bmatrix}0\\ 0\end{bmatrix}(0,\pi_{11})\,\theta^2\begin{bmatrix}0\\ 0\end{bmatrix}(0,\pi_{22})}{\theta^2\begin{bmatrix}0\\ 0\end{bmatrix}(0,\pi_{11})\,\theta^2\begin{bmatrix}1\\ 0\end{bmatrix}(0,\pi_{22})\,\theta^2\begin{bmatrix}0\\ 1\end{bmatrix}(0,\pi_{11})\,\theta^2\begin{bmatrix}1\\ 0\end{bmatrix}(0,\pi_{22})} = \frac{\theta^4\begin{bmatrix}0\\ 0\end{bmatrix}(0,\pi_{22})}{\theta^4\begin{bmatrix}1\\ 0\end{bmatrix}(0,\pi_{22})}.$$

This final quotient equals $1/\lambda$, the fourth branch point on the second surface S_2 obtained in the degeneration of S. Thus, since all of the theta functions are analytic and not zero, we find that the differences $(1/\lambda_1) - (1/\lambda_3)$ and $(1/\lambda_1) - (1/\lambda_2)$ are related by a constant complex multiplicative factor plus terms which are small for π_{12} small. Indeed, we may write

$$\frac{\theta^2\begin{bmatrix}00\\10\end{bmatrix}\theta^2\begin{bmatrix}00\\00\end{bmatrix}}{\theta^2\begin{bmatrix}01\\00\end{bmatrix}\theta^2\begin{bmatrix}01\\10\end{bmatrix}} = \frac{1}{\lambda}(1 + C_1\pi_{12}{}^2 + C_2\pi_{12}{}^4 + \cdots)$$

where

$$C_1 = \left(\frac{\theta''\begin{bmatrix}00\\10\end{bmatrix}}{\theta\begin{bmatrix}0\\1\end{bmatrix}(\pi_{11})} + \frac{\theta''\begin{bmatrix}00\\00\end{bmatrix}}{\theta\begin{bmatrix}0\\0\end{bmatrix}(\pi_{11})}\right)\frac{1}{\theta\begin{bmatrix}0\\0\end{bmatrix}(\pi_{22})} - \left(\frac{\theta''\begin{bmatrix}01\\00\end{bmatrix}}{\theta\begin{bmatrix}0\\0\end{bmatrix}(\pi_{11})} + \frac{\theta''\begin{bmatrix}01\\10\end{bmatrix}}{\theta\begin{bmatrix}0\\1\end{bmatrix}(\pi_{11})}\right)\frac{1}{\theta\begin{bmatrix}1\\0\end{bmatrix}(\pi_{22})}$$

and

$$4!C_2 = 6C_1^2 + \frac{2}{\theta\begin{bmatrix}0\\0\end{bmatrix}(\pi_{11})\,\theta\begin{bmatrix}0\\1\end{bmatrix}(\pi_{11})} \times$$

$$\Bigg(\Bigg(\theta^{(\mathrm{IV})}\begin{bmatrix}00\\10\end{bmatrix}\theta\begin{bmatrix}0\\0\end{bmatrix}(\pi_{11})\,\theta\begin{bmatrix}0\\0\end{bmatrix}(\pi_{22}) + 6\theta''\begin{bmatrix}00\\10\end{bmatrix}\theta''\begin{bmatrix}00\\00\end{bmatrix}$$

$$+\ \theta^{(\mathrm{IV})}\begin{bmatrix}00\\00\end{bmatrix}\theta\begin{bmatrix}0\\1\end{bmatrix}(\pi_{11})\,\theta\begin{bmatrix}0\\0\end{bmatrix}(\pi_{22})\Bigg)\frac{1}{\theta^2\begin{bmatrix}0\\0\end{bmatrix}(\pi_{22})}$$

$$-\ \Bigg(\theta^{(\mathrm{IV})}\begin{bmatrix}01\\00\end{bmatrix}\theta\begin{bmatrix}0\\1\end{bmatrix}(\pi_{11})\theta\begin{bmatrix}1\\0\end{bmatrix}(\pi_{22}) + 6\theta''\begin{bmatrix}01\\00\end{bmatrix}\theta''\begin{bmatrix}01\\10\end{bmatrix}$$

$$+\ \theta^{(\mathrm{IV})}\begin{bmatrix}01\\10\end{bmatrix}\theta\begin{bmatrix}0\\0\end{bmatrix}(\pi_{11})\,\theta\begin{bmatrix}1\\0\end{bmatrix}(\pi_{22})\Bigg)\frac{1}{\theta^2\begin{bmatrix}1\\0\end{bmatrix}(\pi_{22})}\Bigg)$$

where all derivatives are evaluated at $\pi_{12} = 0$.

Using the various entries in Table IV, we obtain all of Theorem 3.

It is interesting to observe that in the degeneration given by Definition 9, we have the limiting value to which the three branch points tend is a point which depends only on π_{11}, i.e., is related to the first limiting surface, while the angles between the three branch points depend on π_{22}, i.e., only the second limiting surface.

5. The second limiting surface

Consider the conformal map $z' = az + b$, where $a = \lambda_1\lambda_2/(\lambda_1 - \lambda_2)$ and $b = -a/\lambda_1$. This map takes $1/\lambda_1$ to $0' = (1/\lambda_1)'$, $1/\lambda_2$ to $1' = (1/\lambda_2)'$ and $1/\lambda_3$ to

$$\left[\left[\frac{\lambda_1-\lambda_3}{\lambda_1-\lambda_2}\right] \frac{\lambda_2}{\lambda_3}\right]' = \left[\frac{1}{\lambda_3}\right]'.$$

These three points together with ∞ are the branch points on the second Riemann surface, S_2.

Let S degenerate as in Definition 9. The branch points 0 and 1, of S, go off to ∞ under the conformal map, that is, 0, 1 and ∞ coalesce, while $(1/\lambda_3)'$ becomes, using Table IV,

$$\frac{\theta^4\begin{bmatrix}0\\0\end{bmatrix}(0,\pi_{22})}{\theta^4\begin{bmatrix}1\\0\end{bmatrix}(0,\pi_{22})}$$

The bounding cycle on S which separates S_1 and S_2 is the cycle surrounding the three "moveable" branch points, $1/\lambda_1$, $1/\lambda_2$, $1/\lambda_3$.

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